On the L^p-estimates of Riesz transforms on forms over complete Riemanian manifolds

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Abstract

In our previous paper [4], we proved a martingale transform representation formula for the Riesz transforms on forms over complete Riemannian manifolds, and proved some explicit L^p-norm estimates for the Riesz transforms on complete Riemannian manifolds with suitable curvature conditions. In this paper we correct a gap contained in [4] and prove that the main result obtained in [4] on the L^p-norm estimates for the Riesz transforms on forms remains valid. Moreover, we prove a time reversal martingale transform representation formula for the Riesz transforms on forms. Finally, we extend our approach and result to the Riesz transforms acting on Euclidean vector bundles over complete Riemannian manifolds with suitable curvature conditions.

1 Introduction

In our previous paper [4] (Theorem 5.3 p. 507), we obtained the following martingale transform representation formulas for the Riesz transforms on forms over complete Riemannian manifolds:

$$R_a^1(\Box_\phi)\omega(x) = -2 \lim_{y\to+\infty} E_y\left[\int_0^\tau e^{a(s-\tau)} M_{\tau,k+1} M_{s,k+1}^{-1} dQ_a\omega(X_s, B_s)dB_s \,\middle|\, X_\tau = x\right],$$

$$R_a^2(\Box_\phi)\omega(x) = -2 \lim_{y\to+\infty} E_y\left[\int_0^\tau e^{a(s-\tau)} M_{\tau,k-1} M_{s,k-1}^{-1} d_\phi^* Q_a\omega(X_s, B_s)dB_s \,\middle|\, X_\tau = x\right].$$

where $R_a^1(\Box_{\phi,k}) = d(a+\Box_{\phi,k})^{-1/2}$ and $R_a^2(\Box_{\phi,k}) = d_\phi^*(a+\Box_{\phi,k})^{-1/2}$. Recently, Bañuelos and Baudoin [1] pointed out that, since $e^{-a\tau}M_{\tau,k\pm1}$ are not adapted with respect to the

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filtration $\mathcal{F}_t = \sigma(X_s : s \in [0,t])$ for $t < \tau$, the above representation formulas should be corrected as follows

$$R_a^1(\Box_\phi)\omega(x) = -2 \lim_{y\to+\infty} E_y\left[e^{-a\tau}M_{\tau,k+1}\int_0^\tau e^{as}M_{s,k+1}^{-1}dQ_a\omega(X_s,B_s)dB_s\bigg| X_\tau = x\right], \quad (1)$$

$$R_a^2(\Box_\phi)\omega(x) = -2 \lim_{y\to+\infty} E_y\left[e^{-a\tau}M_{\tau,k-1}\int_0^\tau e^{as}M_{s,k-1}^{-1}d_\phi^*Q_a\omega(X_s,B_s)dB_s\bigg| X_\tau = x\right]. \quad (2)$$

Indeed, a careful check of the original proof of Theorem 5.3 in [4] indicates that the correct probabilistic representation formula of the Riesz transforms $d(a+\Box_\phi)^{-1/2}$ and $d_\phi^*(a+\Box_\phi)^{-1/2}$ should be given by (1) and (2). See Section 2 below. By the above observation, Bañuelos and Baudoin [1] pointed out that there is a gap in the proof of the L^p-norm estimates of the Riesz transforms $d(a+\Box_\phi)^{-1/2}$ and $d_\phi^*(a+\Box_\phi)^{-1/2}$ in [4] and they proved a new martingale inequality which can be used to correct this gap. In this paper, we correct the above gap and prove that our main result obtained in [4] on the L^p-norm estimates of the Riesz transforms on forms remains valid. Moreover, we prove a time reversal martingale transform representation formula for the Riesz transforms on forms. Finally, we extend our approach and result to the Riesz transforms acting on Euclidean vector bundles over complete Riemannian manifolds with suitable curvature conditions.

2 Martingale transform representation formulas

Let (M,g) be a complete Riemannian manifold, ∇ the gradient operator on M, $\Delta = \mathrm{Tr}\nabla^2$ the covariant Laplace-Beltrami operator on M. Let $\phi \in C^2(M)$, $L = \Delta - \nabla\phi\cdot\nabla$, and $d\mu = e^{-\phi}dv$, where dv is the standard Riemannian volume measure on M.

Let d be the exterior differential operator, d_ϕ^* be its L^2-adjoint with respect to the weighted volume measure $d\mu = e^{-\phi}dv$. Let W_k be the Weitzenböck curvature operator acting on k-forms, and $d\Lambda^k\nabla^2\phi$ be the k-linear endomorphism induced by $\nabla^2\phi$ on Λ^kT^*M. Let $\Box_\phi = dd_\phi^* + d_\phi^*d$ be the Witten Laplacian acting on forms over (M,g) with respect to the weighted volume measure $d\mu = e^{-\phi}dv$. Recall that the Bochner-Weitzenböck formula reads as

$$\Box_{\phi,k} = -(\Delta - \nabla_{\nabla\phi}) + W_k + d\Lambda^k\nabla^2\phi.$$

For all $\omega \in C_0^\infty(\Lambda^kT^*M)$, the Poisson integral $Q_a\omega(x,y)$, also denoted by $\omega_a(x,y)$, is defined by

$$Q_a\omega(x,y) = e^{-y\sqrt{a+\Box_\phi}}\omega(x), \qquad \forall x\in M, y\geq 0.$$

By [4], the Riesz transforms associated with the Witten Laplacian are defined as follows

$$\begin{aligned} R_a^1(\Box_{\phi,k}) &= d(a+\Box_{\phi,k})^{-1/2}, \\ R_a^2(\Box_{\phi,k}) &= d_\phi^*(a+\Box_{\phi,k})^{-1/2}. \end{aligned}$$

Let B_t be one dimensional Brownian motion on $\mathbb{R}$ starting from $B_0 = y > 0$ and with infinitesimal generator $\frac{1}{2}\frac{d^2}{dy^2}$. Let

$$\tau = \inf\{t > 0 : B_t = 0\}.$$

Let X_t be the L-diffusion process on M. Let W_t be the standard Brownian motion on $\mathbb{R}^n$ such that

$$dX_t = U_t \circ dW_t - \nabla\phi(X_t)dt,$$

where $U_t \in \mathrm{End}(T_{X_0}M, T_{X_t}M)$ denotes the stochastic parallel transport along (X_t). Let $M_{k,t} \in \mathrm{End}(\Lambda^k T^*_{X_0}M, \Lambda^k T^*_{X_t}M)$ be the solution to the following covariant SDE along the trajectory of (X_t):

$$\frac{\nabla M_{t,k}}{\partial t} = -(W_k + d\Lambda^k\nabla^2\phi)(X_t)M_{t,k}, \qquad M_{0,k} = \mathrm{Id}_{\Lambda^k T^*_{X_0}M}.$$

In the particular case where $W_k + d\Lambda^k\nabla^2\phi = -a$, where $a \geq 0$ is a constant, we have

$$M_{t,k} = e^{at}U_t, \qquad \forall t \geq 0.$$

The following results is the correct reformulation of Proposition 5.1 in [4].

Proposition 2.1 *For all $\omega \in C_0^\infty(\Lambda^k T^*M)$ and $a \geq 0$, we have*

$$\omega(X_\tau) = e^{a\tau}M^{*,-1}_{\tau,k}\omega_a(Z_0) + e^{a\tau}M^{*,-1}_{\tau,k}\int_0^\tau e^{-as}M^*_{s,k}\left(\nabla, \frac{\partial}{\partial y}\right)\omega_a(Z_s)\cdot(U_s dW_s, dB_s). \tag{3}$$

Proof. By Itô's calculus, we have (see p.504 in [4])

$$e^{-at}M^*_{t,k}\omega(X_t) = e^{-as}M^*_{s,k}\omega_a(Z_s) + \int_s^t e^{-ar}M^*_{r,k}\left(\nabla, \frac{\partial}{\partial y}\right)\omega_a(Z_r)\cdot(U_r dW_r, dB_r)$$

Taking $s = 0$ and $t = \tau$, we obtain Proposition 2.1. $\square$

The following results is the correct reformulation of Theorem 5.2 in [4].

Theorem 2.2 *Suppose that $W_k + d\Lambda^k\nabla^2\phi \geq -a$, where a is a non-negative constant. Then, for all $\omega \in C_0^\infty(\Lambda^k T^*M)$, we have*

$$\frac{1}{2}\omega(x) = \lim_{y\to\infty} E_y\left[e^{-a\tau}M_{\tau,k}\int_0^\tau e^{as}M^{-1}_{s,k}\frac{\partial}{\partial y}\omega_a(X_s, B_s)dB_s\,\middle|\, X_\tau = x\right]. \tag{4}$$

Proof. The proof is indeed a small modification of the original proof given in [4]. For the completeness of the paper, we give the details here. Let $\eta \in C_0^\infty(\Lambda^k T^*M)$. By (3) we have

$$\eta(X_\tau) = e^{a\tau}M^{*,-1}_\tau\eta_a(Z_0) + e^{a\tau}M^{*,-1}_{\tau,k}\int_0^\tau e^{-as}M^*_s\left(\nabla, \frac{\partial}{\partial y}\right)\eta_a(Z_r)\cdot(U_s dW_s, dB_s).$$

Hence

$$\begin{aligned}
&\int_M \left\langle E_y\left[e^{-a\tau}M_{\tau,k}\int_0^\tau e^{as}M^{-1}_{s,k}\frac{\partial}{\partial y}\omega_a(X_s, B_s)dB_s\,\middle|\, X_\tau = x\right], \eta(x)\right\rangle d\mu(x)\\
&= E_y\left[e^{-a\tau}M_{\tau,k}\int_0^\tau e^{as}M^{-1}_{s,k}\frac{\partial}{\partial y}\omega_a(X_s, B_s)dB_s, \eta(X_\tau)\Big\rangle\right]\\
&= I_1 + I_2,
\end{aligned}$$

where

$$
\begin{aligned}
I_1 &= E_y\left[\left\langle e^{-a\tau}M_{\tau,k}\int_0^\tau e^{as}M_{s,k}^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s)dB_s, e^{a\tau}M_{\tau,k}^{*,-1}\eta_a(X_0,B_0)\right\rangle\right],\\
I_2 &= E_y\left[\left\langle e^{-a\tau}M_{\tau,k}\int_0^\tau e^{as}M_{s,k}^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s)dB_s,\right.\right.\\
&\qquad\left.\left. e^{a\tau}M_{\tau,k}^{*,-1}\int_0^\tau e^{-as}M_{s,k}^{*}(\nabla,\partial_y)\eta_a(X_s,B_s)\cdot(U_sdW_s,dB_s)\right\rangle\right].
\end{aligned}
$$

Using the martingale property of the Itô integral, we have

$$
\begin{aligned}
I_1 &= E_y\left[\left\langle\int_0^\tau e^{as}M_{s,k}^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s)dB_s, \eta_a(X_0,B_0)\right\rangle\right]\\
&= E_y\left[\left\langle E\left[\int_0^\tau e^{as}M_{s,k}^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s)dB_s\middle|(X_0,B_0)\right], \eta_a(X_0,B_0)\right\rangle\right]\\
&= 0.
\end{aligned}
$$

On the other hand, using the L^2-isometry of the Itô integral, we have

$$
\begin{aligned}
I_2 &= E_y\left[\int_0^\tau\left\langle e^{as}M_{s,k}^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s), e^{-as}M_{s,k}^{*}\frac{\partial}{\partial y}\eta_a(X_s,B_s)\right\rangle ds\right]\\
&= E_y\left[\int_0^\tau\left\langle\frac{\partial}{\partial y}\omega_a(X_s,B_s),\frac{\partial}{\partial y}\eta_a(X_s,B_s)\right\rangle ds\right].
\end{aligned}
$$

The Green function of the background radiation process is given by $2(y\wedge z)$. Hence

$$
\begin{aligned}
&E_y\left[\int_0^\tau\left\langle\frac{\partial}{\partial y}\omega_a(X_s,B_s),\frac{\partial}{\partial y}\eta_a(X_s,B_s)\right\rangle ds\right]\\
&\qquad = 2\int_M\int_0^\infty (y\wedge z)\left\langle\frac{\partial}{\partial z}\omega_a(x,z),\frac{\partial}{\partial z}\eta_a(x,z)\right\rangle dzd\mu(x).
\end{aligned}
$$

By spectral decomposition, we have the Littelwood-Paley identity

$$
\lim_{y\to\infty}\int_M\int_0^\infty (y\wedge z)\left\langle\frac{\partial}{\partial z}\omega_a(x,z),\frac{\partial}{\partial z}\eta_a(x,z)\right\rangle dzd\mu(x) = \int_M\langle\omega(x),\eta(x)\rangle d\mu(x).
$$

Thus

$$
\langle\omega,\eta\rangle_{L^2(\mu)} = 2\lim_{y\to\infty}\int_M\left\langle E_y\left[e^{-a\tau}M_{\tau,k}\int_0^\tau e^{as}M_{s,k}^{-1}\frac{\partial}{\partial y}\omega_a(X_s,B_s)dB_s\middle| X_\tau=x\right],\eta(x)\right\rangle d\mu(x).
$$

This completes the proof of Theorem 2.2. □

The following martingale transform representation formulas of the Riesz transforms on k-forms on complete Riemannian manifolds are the correct reformulations of the ones that we obtained in Theorem 5.3 in [4]. In the case $k=0$, see [3, 6].

Theorem 2.3 *Under the above notation, for all $\omega\in C_0^\infty(\Lambda^kT^*M)$, we have*

$$
R_a^1(\Box_\phi)\omega(x) = -2\lim_{y\to+\infty}E_y\left[e^{-a\tau}M_{\tau,k+1}\int_0^\tau e^{as}M_{s,k+1}^{-1}dQ_a\omega(X_s,B_s)dB_s\middle| X_\tau=x\right], \quad (5)
$$

$$R_a^2(\Box_\phi)\omega(x) = -2 \lim_{y\to+\infty} E_y\left[e^{-a\tau}M_{\tau,k-1}\int_0^\tau e^{as}M_{s,k-1}^{-1}d_\phi^* Q_a\omega(X_s,B_s)dB_s\middle| X_\tau = x\right]. \quad (6)$$

In particular, in the case where $W_{k+1} + d\Lambda^{k+1}\nabla^2\phi = -a$*, we have*

$$R_a^1(\Box_\phi)\omega(x) = -2 \lim_{y\to+\infty} E_y\left[U_\tau\int_0^\tau U_s^{-1}dQ_a\omega(X_s,B_s)dB_s\middle| X_\tau = x\right], \quad (7)$$

and in the case where $W_{k-1} + d\Lambda^{k-1}\nabla^2\phi = -a$*, we have*

$$R_a^2(\Box_\phi\omega(x) = -2 \lim_{y\to+\infty} E_y\left[U_\tau\int_0^\tau U_s^{-1}d_\phi^* Q_a\omega(X_s,B_s)dB_s\middle| X_\tau = x\right]. \quad (8)$$

Proof. The proof is as the same as the one of Theorem 5.3 in [4]. Indeed, applying Theorem 2.2 to $R_a^1(\Box_\phi)\omega = d(a+\Box_{\phi,k})^{-1/2}\omega$, we have

$$\begin{aligned} -\frac{1}{2}R_a^1(\Box_\phi)\omega(x) \quad &= \quad \lim_{y\to\infty} E_y\Bigg[e^{-a\tau}M_{\tau,k+1}\int_0^\tau e^{as}M_{s,k+1}^{-1}\sqrt{a+\Box_{\phi,k+1}} \\ &\qquad Q_{k+1,a}d(a+\Box_{\phi,k})^{-1/2}\omega_a(X_s,B_s)dB_s\Bigg| X_\tau = x\Bigg]. \end{aligned}$$

Using the commutation formula

$$d\sqrt{a+\Box_{\phi,k}}\omega = \sqrt{a+\Box_{\phi,k+1}}d\omega,$$

we obtain

$$-\frac{1}{2}R_a^1(\Box_\phi)\omega(x) = \lim_{y\to\infty} E_y\left[e^{-a\tau}M_{\tau,k+1}\int_0^\tau e^{as}M_{s,k+1}^{-1}dQ_{k,a}\omega_a(X_s,B_s)dB_s\middle| X_\tau = x\right].$$

This proves (5). Similarly, we can prove (6). Note that, if $W_{k\pm1} + \nabla^2\phi = -a$, we have $M_{t,k\pm1} = e^{at}U_t$ for all $t \geq 0$. Thus, (7) (resp. (8)) follows from (5) (resp. (6)). $\Box$

Remark 2.4 Similarly we have the following martingale transform representation for the Riesz potential on forms.

$$\frac{1}{2}(a+\Box_\phi)^{-1/2}\omega(x) = -\lim_{y\to\infty}\left[e^{-a\tau}M_{\tau,k}\int_0^\tau e^{as}M_{s,k}^{-1}\omega_a(X_s,B_s)dB_s\middle| X_\tau = x\right].$$

In particular, under the condition $W_k + d\Lambda^k\nabla^2\phi \geq 0$, we have

$$\frac{1}{2}\Box_\phi^{-1/2}\omega(x) = -\lim_{y\to\infty}\left[M_{\tau,k}\int_0^\tau M_{s,k}^{-1}\omega(X_s,B_s)dB_s\middle| X_\tau = x\right],$$

where $\omega(x,y) = e^{-y\sqrt{\Box_\phi}}\omega(x)$ denotes the Poisson semigroup generated by $\Box_\phi$ on $L^2(\Lambda^k T^*M,\mu)$.

3 The L^p-norm estimate

In this section we correct a gap contained in [4] and prove that our main result obtained in [4] on the L^p-norm estimates of the Riesz transforms on forms remains valid. When $p=2$, we have the following

Proposition 3.1 *For all $a \geq 0$ and $\omega \in C_0^\infty(\Lambda^k T^*M)$, we have*

$$\begin{aligned}\|d(a+\Box_\phi)^{-1/2}\omega\|_2 &\leq \|\omega\|_2,\\ \|d_\phi^*(a+\Box_\phi)^{-1/2}\omega\|_2 &\leq \|\omega\|_2.\end{aligned}$$

Proof. By Gaffney's integration by part, for all $\omega \in C_0^\infty(\Lambda^k T^*M)$, we have

$$\langle\langle (a+\Box_\phi)\omega, \omega\rangle\rangle = a\|\omega\|_2^2 + \|d\omega\|_2^2 + \|d_\phi^*\omega\|_2^2.$$

Since $a + \Box_\phi$ is non-negative symmetric operator on $L^2(\Lambda^k T^*M, \mu)$, we get

$$\|d\omega\|_2^2 + \|d_\phi^*\omega\|_2^2 + a\|\omega\|_2^2 = \|\sqrt{a+\Box_\phi}\omega\|_2^2.$$

This implies that

$$\|d(a+\Box_\phi)^{-1/2}\omega\|_2^2 + \|d_\phi^*(a+\Box_\phi)^{-1/2}\omega\|_2^2 \leq \|\omega\|_2^2.$$

The proof of Proposition 3.1 is completed. $\Box$

The following result is the restatement of the main result (i.e., Theorem 1.6) in [4].

Theorem 3.2 *Let M be a complete Riemannian manifold, and $\phi \in C^2(M)$. Suppose that there exists a constant $a \geq 0$ such that*

$$W_k + d\Lambda^k \nabla^2\phi \geq -a, \quad \text{and} \quad W_{k+1} + d\Lambda^{k+1}\nabla^2\phi \geq -a.$$

Then, there exists a constant $C_k > 0$ depending only on k such that for all $p > 1$,

$$\|d(a+\Box_\phi)^{-1/2}\omega\|_p \leq C_k(p^*-1)^{-3/2}\|\omega\|_p, \quad \forall \omega \in C_0^\infty(\Lambda^k T^*M). \tag{9}$$

In particular, if $W_k + d\Lambda^k\nabla^2\phi \geq 0$ and $W_{k+1} + d\Lambda^{k+1}\nabla^2\phi \geq 0$, then the Riesz transform $d\Box_\phi^{-1/2}$ is bounded in L^p for all $p > 1$, and there exists a constant $C_k > 0$ depending only on k such that for all $p > 1$,

$$\|d\Box_\phi^{-1/2}\omega\|_p \leq C_k(p^*-1)^{-3/2}\|\omega\|_p, \quad \forall \omega \in C_0^\infty(\Lambda^k T^*M). \tag{10}$$

Proof. By Theorem 2.3, Fatou's lemma, and using the L^p-contractivity of the conditional expectation, for any $1 < p < \infty$, we have

$$\begin{aligned}&\|d(a+\Box_\phi)^{-1/2}\omega\|_p^p\\ =\;& 2^p\int_M \lim_{y\to\infty}\left|E_y\left[e^{-a\tau}M_{\tau,k+1}\int_0^\tau e^{as}M_{s,k+1}^{-1}dQ_{a,k}\omega(X_s,B_s)dB_s\,\middle|\, X_\tau = x\right]\right|^p d\mu(x)\\ \leq\;& 2^p\liminf_{y\to\infty}\int_M E_y\left[\left|e^{-a\tau}M_{\tau,k+1}\int_0^\tau e^{as}M_{s,k+1}^{-1}dQ_{a,k}\omega(X_s,B_s)dB_s\right|^p \,\middle|\, X_\tau = x\right]d\mu(x)\\ =\;& 2^p\liminf_{y\to\infty} E_y\left[\left|e^{-a\tau}M_{\tau,k+1}\int_0^\tau e^{as}M_{s,k+1}^{-1}dQ_{a,k}\omega(X_s,B_s)dB_s\right|^p\right].\end{aligned}$$

Recall that, see p. 509-p. 510 in [4], there is an $(n+1)\times(n+1)$ operator valued matrix such that

$$d\omega(x,y) = A\overline{\nabla}\omega(x,y),$$

where $\overline{\nabla} = (\nabla, \partial_y)$. Moreover, $\|A\|_{\rm op}$ is a finite number depending only on k. In view of this, we have

$$\|d(a+\Box_\phi)^{-1/2}\omega\|_p \le 2\lim\inf_{y\to\infty}\left\|e^{-a\tau}M_{\tau,k+1}\int_0^\tau e^{as}M_{s,k+1}^{-1}A\overline{\nabla}Q_{a,k}\omega(X_s,B_s)\cdot(U_sdW_s,dB_s)\right\|_p. \quad (11)$$

Let

$$I_y = e^{-a\tau}M_{\tau,k+1}\int_0^\tau e^{as}M_{s,k+1}^{-1}A\overline{\nabla}Q_{a,k}\omega(X_s,B_s)\cdot(U_sdW_s,dB_s),$$

and

$$J_y = \left\{\int_0^\tau |\overline{\nabla}Q_{a,k}\omega(X_s,B_s)|^2ds\right\}^{1/2}.$$

By Theorem 2.6 due to Bañuelos and Baudoin in [1], under the assumption $W_k + d\Lambda^k\nabla^2\phi \ge -a$, we can prove that

$$\|I_y\|_p \le 3\sqrt{p(2p-1)}\|A\|_{\rm op}\|J_y\|_p. \quad (12)$$

Moreover, by Proposition 6.2 in our previous paper [4], we have

$$\|J_y\|_p \le B_p\|\omega\|_p,$$

where $B_p = (2p)^{1/2}(p-1)^{-3/2}$ for $p\in(1,2)$, $B_p = 1$ for $p=2$, and $B_p = \frac{p}{\sqrt{2(p-2)}}$ if $p>2$. Combining this with (12), for all $1<p<2$, we can prove that

$$\begin{aligned}\|R_a^1(\Box_\phi)\omega\|_p &\le 2\lim\inf_{y\to\infty}\|I_y\|_p\\ &\le 6\sqrt{2}\|A\|_{\rm op}p(2p-1)^{1/2}(p-1)^{-3/2}\|\omega\|_p\\ &\le 12\sqrt{6}\|A\|_{\rm op}(p-1)^{-3/2}\|\omega\|_p,\end{aligned}$$

and for $p>2$, we have

$$\begin{aligned}\|R_a^1(\Box_\phi)\omega\|_p &\le 2\lim\inf_{y\to\infty}\|I_y\|_p\\ &\le 3\sqrt{2}\|A\|_{\rm op}p^{3/2}(2p-1)^{1/2}(p-2)^{-1/2}\|\omega\|_p\\ &\le 6\|A\|_{\rm op}(p-1)^{3/2}(1+O(1/p))\|\omega\|_p.\end{aligned}$$

This implies the desired L^p-norm estimate for the Riesz transform $d(a+\Box_\phi)^{-1/2}$. □

Remark 3.3 The above proof corrects a gap in the proof of Theorem 1.6 in [4] (p. 510 line 4 to line 5 in [4]), where we used the Burkholder-Davies-Gundy inequality to derive that

$$\|I_y\|_p \le C_p\left\|\left\{\int_0^\tau |e^{a(s-\tau)}M_{\tau,k+1}M_{s,k+1}^{-1}A|^2|\overline{\nabla}Q_{a,k}\omega(X_s,B_s)|^2ds\right\}^{1/2}\right\|_p,$$

where C_p is a constant depending only on p. However, as $e^{-a\tau}M_{\tau,k\pm1}$ are not adapted with respect to the filtration $\mathcal{F}_t = \sigma(X_s : s\in[0,t])$ for $t<\tau$, one cannot use the Burkholder-Davis-Gundy inequality in the above way, except that $e^{-a\tau}M_{\tau,k\pm1}$ is independent of $(X_s, s\in[0,\tau])$, which only happens in the case where $W_{k+1}+d\Lambda^{k+1}\nabla^2\phi \equiv -a$.

Theorem 3.4 *Let M be a complete Riemannian manifold, and $\phi \in C^2(M)$. Suppose that there exists a constant $a \geq 0$ such that*

$$W_k + d\Lambda^k \nabla^2 \phi \geq -a, \quad \text{and} \quad W_{k-1} + d\Lambda^{k-1} \nabla^2 \phi \geq -a.$$

Then, there exists a constant $C_k > 0$ depending only on k such that for all $p > 1$,

$$\|d_\phi^*(a + \Box_\phi)^{-1/2}\omega\|_p \leq C_k(p^* - 1)^{3/2}\|\omega\|_p, \quad \forall \omega \in C_0^\infty(\Lambda^k T^* M). \tag{13}$$

In particular, if $W_k + d\Lambda^k \nabla^2 \phi \geq 0$ and $W_{k-1} + d\Lambda^{k-1} \nabla^2 \phi \geq 0$, then the Riesz transform $d\Box_\phi^{-1/2}$ is bounded in L^p for all $p > 1$. More precisely, there exists a constant $C_k > 0$ depending only on k such that for all $p > 1$,

$$\|d_\phi^* \Box_\phi^{-1/2}\omega\|_p \leq C_k(p^* - 1)^{3/2}\|\omega\|_p, \quad \forall \omega \in C_0^\infty(\Lambda^k T^* M). \tag{14}$$

Proof. By duality argument as used in [4], we can derive Theorem 3.4 from Theorem 3.2. □

4 Case of constant curvature

In the particular case where $W_{k+1} + d\Lambda^{k+1} \nabla^2 \phi \equiv -a$, we have

$$d(a + \Box_\phi)^{-1/2}\omega(x) = -2 \lim_{y \to +\infty} E_y \left[U_\tau \int_0^\tau U_s^{-1} A \overline{\nabla} Q_{a,k}\omega(X_s, B_s) dB_s \middle| X_\tau = x \right].$$

By the same argument as used in the proof of (11),

$$\|d(a + \Box_\phi)^{-1/2}\omega\|_p \leq 2 \lim \inf_{y \to \infty} \left\| U_\tau \int_0^\tau U_s^{-1} A \overline{\nabla} Q_{a,k}\omega(X_s, B_s) \cdot (U_S dW_s, dB_s) \right\|_p.$$

Hence

$$\|d(a + \Box_\phi)^{-1/2}\omega\|_p \leq 2 \lim \inf_{y \to \infty} \left\| \int_0^\tau U_s^{-1} A \overline{\nabla} Q_{a,k}\omega(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p.$$

By Burkholder's sharp L^p-inequality for subordination of martingale transforms [2] we have

$$\|d(a + \Box_\phi)^{-1/2}\omega\|_p \leq 2\|A\|_{\text{op}}(p^* - 1) \lim \inf_{y \to \infty} \left\| \int_0^\tau U_s^{-1} \overline{\nabla} Q_{a,k}\omega(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p.$$

As was pointed out in Remark 6.5 in [4], only if $W_k + d\Lambda^k \nabla^2 \phi = -a$, we can obtain

$$\left\| \int_0^\tau U_s^{-1} \overline{\nabla} Q_{a,k}\omega(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p \leq \|\omega\|_p.$$

That is to say, only if $W_k + d\Lambda^k \nabla^2 \phi = -a$ and $W_{k+1} + d\Lambda^{k+1} \nabla^2 \phi = -a$, which happens in the case where M is a flat Riemannian manifold and $\nabla^2 \phi \equiv 0$, hence $a = 0$, we can obtain $\|d\Box_\phi^{-1/2}\omega\|_p \leq 2\|A\|_{\text{op}}(p^* - 1)\|\omega\|_p$.

Remark 4.1 In view of Theorem 3.2, for all $p > 1$, the upper bound $C_k(p^* - 1)^{3/2}$ appeared in Theorem 1.6 in [4] remains valid, but the upper bound $C_k(p^* - 1)$ appeared in Theorem 1.7 and Theorem 1.8 in [4] should be replaced by $C_k(p^* - 1)^{3/2}$.

5 Time reversal martingale transformation representation formula for the Riesz transfroms

In this section, we prove a time reversal martingale transformation representation formula for the Riesz transforms on forms on complete Riemannian manifolds.

First, we have the following time reversal martingale transformation representation formula for forms.

Theorem 5.1 *Let $\widehat{X}_t = X_{\tau-t}$, and $\widehat{B}_t = B_{\tau-t}$, $t \in [0,\tau]$. Let $\widehat{M}_{t,k}$ be the solution to the covariant SDE*

$$\begin{aligned} \frac{\nabla}{\partial t}\widehat{M}_{t,k} &= -\widehat{M}_t(W_k + d\Lambda^k\nabla^2\phi)(\widehat{X}_t), \\ \widehat{M}_{0,k} &= \mathrm{Id}_{\Lambda^k T^*_{\widehat{X}_0}M}. \end{aligned}$$

For any $\omega \in C_0^\infty(\Lambda^k T^*M)$, let $\omega_a(x,y) = e^{-y\sqrt{a+\Box_\phi}}\omega(x)$, $\forall x \in M, y \geq 0$. Then, for a.s. $x \in M$,

$$\frac{1}{2}\omega(x) = \lim_{y\to+\infty} E_y\left[\widehat{Z}_\tau \middle| \widehat{X}_0 = x\right],$$

where

$$\widehat{Z}_\tau = \int_0^\tau e^{-at}\widehat{M_{t,k}}\partial_y\omega_a(\widehat{X}_t, \widehat{B}_t)d\widehat{B}_t - \int_0^\tau e^{-at}\widehat{M}_{t,k}\partial_y^2\omega(\widehat{X}_t, \widehat{B}_t)dt.$$

Proof. The proof is similarly to the one of Theorem 5.1 in [6]. □

By Theorem 5.1, we can prove the following time reversal martingale transformation representation formula for the Riesz transforms on complete Riemannian manifolds.

Theorem 5.2 *For any $\omega \in C_0^\infty(M, \Lambda^k T^*M)$, we have*

$$\begin{aligned} R_a^1(\Box_{\phi,k})\omega(x) &= -2\lim_{y\to+\infty} E_y\left[\widehat{Z}_{\tau,k+1} \middle| \widehat{X}_0 = x\right], \\ R_a^2(\Box_{\phi,k})\omega(x) &= -2\lim_{y\to+\infty} E_y\left[\widetilde{Z}_{\tau,k-1} \middle| \widehat{X}_0 = x\right], \end{aligned}$$

where

$$\begin{aligned} \widehat{Z}_{\tau,k+1} &= \int_0^\tau e^{-as}\widehat{M}_{s,k+1}dQ_{k,a}\omega(\widehat{X}_s, \widehat{B}_s)d\widehat{B}_s - \int_0^\tau e^{-as}\widehat{M}_{s,k+1}\partial_y dQ_{k,a}\omega(\widehat{X}_s, \widehat{B}_s)ds, \\ \widetilde{Z}_{\tau,k-1} &= \int_0^\tau e^{-as}\widehat{M}_{s,k-1}d_\phi^* Q_{k,a}\omega(\widehat{X}_s, \widehat{B}_s)d\widehat{B}_s - \int_0^\tau e^{-as}\widehat{M}_{s,k-1}\partial_y d_\phi^* Q_{k,a}\omega(\widehat{X}_s, \widehat{B}_s)ds. \end{aligned}$$

Remark 5.3 As noticed in [3], there exists a standard one dimensional Brownian motion β_t such that

$$d\widehat{B}_t = d\beta_t + \frac{dt}{\widehat{B}_t}, \quad t \in (0,\tau].$$

6 Riesz transforms on Euclidean vector bundles

In this section we extend our approach and result to the Riesz transforms acting on Euclidean vector bundles over complete Riemannian manifolds.

Let M be a complete Riemannian manifold, E a Riemannian vector bundle over M. Let ∇^E be a metric preserving connection on E. Let $F = \Lambda^{\cdot} T^*M \otimes E$, and define

$$\nabla^F = \nabla^{\Lambda^{\cdot} T^*M} \otimes 1_E + 1_{\Lambda^{\cdot} T^*M} \otimes \nabla^E.$$

The De Rham operator acting on $C^\infty(M, F)$ is defined by

$$d^F = \sum_{i=1}^{n} e_i^* \wedge \nabla^F_{e_i},$$

where $(e_1, \dots, e_n)$ is a orthonormal basis at any point $x \in M$, and $(e_1^*, \dots, e_n^*)$ is its dual.

The curvature of ∇^E is defined by

$$R^E = (\nabla^E)^2.$$

Suppose that E is an Euclidean vector bundle with flat connection, i.e, $R^E = 0$. Then

$$(d^F)^2 = 0.$$

Let $\phi \in C^2(M)$, $\mu = e^{-\phi}dv$. Let d_ϕ^{F*} be the L^2-adjoint of d^F with respect to μ. We have

$$(d_\phi^{F*})^2 = 0.$$

The Witten Laplacian acting on $C_0^\infty(M, F)$ is defined by

$$\square_{F,\phi} = d^F d_\phi^{F*} + d_\phi^{F*} d^F.$$

The heat semigroup and the Poisson semigroup generated by $\square_{F,\phi}$ are denoted by $P_t\omega(x) = e^{-t\square_{F,\phi}}\omega(x)$ and $Q_a\omega(x, y) = e^{-y\sqrt{a+\square_{F,\phi}}}\omega(x)$ respectively. The Bochner-Weitzenböck formula holds

$$\square_{F,\phi} = -\Delta_{F,\phi} + W_{F,\phi},$$

where $\Delta_{F,\phi} = \mathrm{Tr}(\nabla^F)^2 - \nabla^F_{\nabla\phi}$, and $W_{F,\phi} = W + d\Lambda^{\cdot}\nabla^2\phi$.

Let X_t be the L-diffusion process on M. Let $M_{k,t} \in \mathrm{End}(\Lambda^k T^*_{X_0} M \otimes E, \Lambda^k T^*_{X_t} M \otimes E)$ be the solution to the following covariant SDE along the trajectory of (X_t):

$$\frac{\nabla M_{t,k}}{\partial t} = -(W_k + d\Lambda^k \nabla^2\phi)(X_t) M_{t,k}, \qquad M_{0,k} = \mathrm{Id}_{\Lambda^k T^*_{X_0} M \otimes E}.$$

We have the following results on the quantitative L^p-estimates of the Riesz transforms on Euclidean vector bundles over complete Riemannian manifolds.

Theorem 6.1 *Let M be a complete Riemannian manifold, E be an Euclidean vector bundle over M, and $\phi \in C^2(M)$. Then, for all $\omega \in C_0^\infty(M, \Lambda^k T^*M \otimes E)$ and for all μ-a.s. $x \in M$, we have*

$$d^F(a + \square_{F,\phi})^{-1/2}\omega(x) = -2 \lim_{y\to\infty} E_y \left[e^{-a\tau} M_{\tau,k+1} \int_0^\tau e^{as} M_{s,k+1}^{-1} d^F Q_a\omega(X_s, B_s) dB_s \,\middle|\, X_\tau = x \right].$$

Suppose that $W_i + d\Lambda^i\nabla^2\phi \geq -a$, $i = k,\ k+1$. Then, for all $p > 1$ and for all $\omega \in L^p(\Lambda^k T^*M \otimes E, \mu)$, we have

$$\|d^F(a + \Box_{F,\phi})^{-1/2}\omega\| \leq C_p\|A\|\|\omega\|_p$$

where $\|A\|$ is the operator norm of $A \in \mathrm{End}(F, F)$ is such that $d^F\omega = A\nabla\omega$ and depends only on k, C_p is a constant depending only on p, more precisely, $C_p = \|A\|^{-1}$ for $p = 2$, $C_p = O(p^* - 1)^{3/2}$ for $p \to 1$ and $p \to \infty$.

Proof. The proof is as the same as the one of Theorem 2.3 and Theorem 3.2. □

Theorem 6.2 *Let M be a complete Riemannian manifold, E be an Euclidean vector bundle over M, and $\phi \in C^2(M)$. Then, for all $\omega \in C_0^\infty(M, \Lambda^k T^*M \otimes E)$ and for all μ-a.s. $x \in M$, we have*

$$d_\phi^{F*}(a + \Box_{F,\phi})^{-1/2}\omega(x) = -2\lim_{y\to\infty} E_y\left[e^{-a\tau}M_{\tau,k-1}\int_0^\tau e^{as}M_{s,k-1}^{-1}d_\phi^{F*}Q_a\omega(X_s, B_s)dB_s\middle| X_\tau = x\right],$$

Suppose that $W_i + d\Lambda^i\nabla^2\phi \geq -a$, $i = k,\ k-1$. Then, for all $p > 1$ and for all $\omega \in L^p(\Lambda^k T^*M \otimes E, \mu)$, we have

$$\|d_\phi^{F*}(a + \Box_{F,\phi})^{-1/2}\omega\| \leq C_p\|A\|\|\omega\|_p$$

where C_p is a constant depending only on p, more precisely, $C_p = \|A\|^{-1}$ for $p = 2$, and $C_p = O(p^* - 1)^{3/2}$ for $p \to 1$ and $p \to \infty$.

Proof. By duality argument, we can derive Theorem 6.2 from Theorem 6.1. □

To end this paper, let us mention that, in a forthcoming paper [7], we will prove a martingale transform representation formula for the Riesz transforms associated with the Dirac operator acting on Hermitian vector bundles over complete Riemannian manifolds and for the Riesz transforms associated with the $\bar{\partial}$-operator acting on holomorphic Hermitian vector bundles over complete Kähler manifolds. By the same argument as used in this paper and in [6], we can prove some explicit dimension free L^p-norm estimates of these Riesz transforms on complete Riemannian or Kähler manifolds with suitable curvature conditions. See also [5].

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